Circuit Synthesis of 3-D Rotation Orthonormalization

Jin Wu, *Member, IEEE*, Yi Jiang, *Member, IEEE*, Chong Li, Ran Sun, Chengxi Zhang, *Member, IEEE*, Yang Yu, Yilong Zhu and Ming Liu, *Senior Member, IEEE*

***Abstract*—In many pose estimation problems, rotation matrices can not always be estimated subject to all nonlinear rigidity constraints. Therefore, engineers tend to obtain the nearest rotation matrix of an improper one, which is called the rotation orthonormalization problem. In this paper, we show a circuit synthesis of such problem by using only simple algebraic components. We give theoretical convergence analysis of the designed circuit. By using the proposed circuit, rotation orthonormalization can be easily performed without the need of previous sophisticated processes like singular value decomposition (SVD) and eigen-decomposition (EIG). Experiments of the developed method's characteristics are conducted. The circuitized scheme has also been implemented on an FPGA platform.**

***Index Terms*—Rotation orthonormalization, circuit synthesis, algebraic circuits, pose estimation, 3-D registration**

I. Introduction

A. Background

ROTATION matrices in 3-D space are useful in many applications involving pose estimation. In the field of computer vision, estimating camera poses from visual measurements can provide autonomous perception of robots. By introducing inertial sensing technology, object poses can be obtained by inertial integration mechanisms [1], [2]. In these approaches, all rotation matrices must be proper, i.e. they belong to the special orthogonal group $SO(3)$ such that $SO(3) := \left\{\boldsymbol{R} \in \mathbb{R}^{3\times 3} | \boldsymbol{R}^\top \boldsymbol{R} = \boldsymbol{I}, \det(\boldsymbol{R}) = 1\right\}$. Namely, one rotation matrix is an orthogonal one with a positive determinant, such is the rigidity nature of rotation. In many algorithms for pose estimation, due to the nonlinear constraints of $\boldsymbol{R}$, one may usually obtain an improperly orthonormalized rotation and then find the nearest rotation to it. For instance, in the problem of perspective-n-point (PnP) [3], given 2-D image points in the image plane $\boldsymbol{p}$ and corresponding 3-D coordinates in the world frame $\boldsymbol{v}$, the rotation $\boldsymbol{R} \in SO(3)$ and translation $\boldsymbol{t} \in \mathbb{R}^3$ satisfy the following equality

$$s\left(\boldsymbol{u}^\top, 1\right)^\top = \boldsymbol{K}\left(\boldsymbol{R}\boldsymbol{p} + \boldsymbol{t}\right) \tag{1}$$

in which s denotes the scale factor which is a function of $\boldsymbol{R}$ and $\boldsymbol{t}$, while $\boldsymbol{K}$ is the camera intrinsic matrix consisting of focal length and principal point. The direct linear transform (DLT) seeks an approximate $\boldsymbol{R}$ by vectorizing (1). According to the noise in $\boldsymbol{p}$ and $\boldsymbol{v}$, $\boldsymbol{R}$ should be projected on $SO(3)$, which is the kernel problem we discuss in this paper, i.e. rotation orthonormalization. This aspect is so frequent that, in almost all pose estimation problems, we require rotation orthonormalization. This is not only because of the use of algorithm with improper rotation estimation but also due to the fact that, in most cases rotation parameters are stored with limited word-length. Thus, reading them from files may require re-orthonormalization to guarantee rigidity.

Rotation orthonormalization is not an easy problem for all computational platforms. The reason is that conventional works mainly consider singular value decomposition (SVD) and eigen-decomposition (EIG), which are sophisticated for platforms without efficient floating-number supports, e.g. low-cost microcontrollers, field-programmable gate array (FPGA) and graphics processing unit (GPU). Most floating-number operations of modern computers are accelerated thanks to the invention of the float point unit (FPU). However, it does not appear so frequently in parallel platforms due to bus parallelization problems. Therefore, an efficient rotation orthonormalization with simple algebraic steps will be required. Mentioned below are some representations regarding the previous development of this problem.

B. Related Work

Rotation orthonormalization is a classical problem, starting from inertial integration of rotation using inertial measurements:

$$\dot{\boldsymbol{R}} = \boldsymbol{R}\boldsymbol{\omega}_\times \tag{2}$$

in which $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^\top$ is the angular-rate vector, and $\boldsymbol{\omega}_\times$ is the skew-symmetric form of $\boldsymbol{\omega}$. Integrating (2) over time will gradually generate an improper rotation sequence that needs to be further orthonormalized. Bar-Itzhack et al.

Manuscript received X X, 20XX; revised X X, 20XX; accepted X X, 20XX. This research was supported by Shenzhen Science, Technology and Innovation Comission (SZSTI) JCYJ20160401100022706, in part by General Research Fund of Research Grants Council Hong Kong, 11210017, in part by Early Career Scheme Project of Research Grants Council Hong Kong, 21202816. This brief was recommended by X X. (*Corresponding authors: Chong Li, Yi Jiang*)

J. Wu, Y. Yang, Y. Zhu and M. Liu are with Department of Electronic and Computer Engineering, Hong Kong University of Science and Technology, Hong Kong, China. (e-mail: eelium@ust.hk)

Y. Jiang is with the State Key Laboratory of Synthetical Automation for Process Industries and International Joint Research Laboratory of Integrated Automation, Northeastern University, Shenyang, China. (e-mail: JY369356904@163.com)

C. Li is with Department of Automation and Measurement, Ocean University of China. (e-mail: czl0047@auburn.edu)

R. Sun is with School of Materials Science and Engineering, Shanghai Jiao Tong University, Shanghai, China. (e-mail: sunr1990@sjtu.edu.cn)

C. Zhang is with School of Electronic and Information Engineering, Harbin Institute of Technology, Shenzhen, China. (e-mail: dongfangxy@163.com)

Color versions of one or more of the figures in this brief are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TCSII.2020.XXXX

proposed the first solution in [4] by conducting iteration with indices $n = 1, 2, \cdots$ for orthonormalizing a noisy rotation matrix $\boldsymbol{B}$ with initial condition of $\boldsymbol{D}_0 = \boldsymbol{B}$, such that $\boldsymbol{D}_n = \mathrm{Norm}\left[\left(\mathrm{adj}\boldsymbol{D}_{n-1} + \boldsymbol{D}_{n-1}\right)/2\right]$, where adj denotes the adjoint matrix and the operation Norm denotes normalizing the matrix in terms of all rows. Rotation orthonormalization is actually the essence of the 3-D registration problem [5], provided that two point clouds $\boldsymbol{b}_i \in \{\mathcal{B}\}$ and $\boldsymbol{r}_i \in \{\mathcal{R}\}$ with N point correspondences are to be aligned via

$$\boldsymbol{b}_i = \boldsymbol{R}\boldsymbol{r}_i + \boldsymbol{t}, \qquad i = 1, 2, \cdots, N. \tag{3}$$

Least-square solution of (3) is called the Wahba's problem initialized in 1965. It can be reduced to

$$\mathop{\arg\min}_{\boldsymbol{R} \in SO(3)} \mathrm{tr}\left(\boldsymbol{R}\boldsymbol{B}^\top\right) \tag{4}$$

in which $\boldsymbol{B} = 1/N \sum_{i=1}^{N} \left(\boldsymbol{b}_i - \bar{\boldsymbol{b}}\right)\left(\boldsymbol{r}_i - \bar{\boldsymbol{r}}\right)^\top$ with $\bar{\boldsymbol{b}}$ and $\bar{\boldsymbol{r}}$ being gravitational centers of $\{\mathcal{B}\}$ and $\{\mathcal{R}\}$ respectively and $\boldsymbol{t} = \bar{\boldsymbol{b}} - \boldsymbol{R}\bar{\boldsymbol{r}}$. (4) precisely defines the mathematical framework of rotation orthonormalization. To solve $\boldsymbol{R}$ from (4), SVD can be performed [6], [7] so that $\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top = \boldsymbol{B}$ and $\boldsymbol{R} = \boldsymbol{U}\mathrm{diag}\left[1, 1, \det(\boldsymbol{U}\boldsymbol{V})\right]\boldsymbol{V}^\top$. If we parameterize $\boldsymbol{R}$ using unit quaternion $\boldsymbol{q}$, (4) is equivalent to finding the eigenvector associated with the maximum eigenvalue $\lambda_{\max}$ of

$$\boldsymbol{W} = \begin{bmatrix} \mathrm{tr}(\boldsymbol{B}) & \boldsymbol{z}^\top \\ \boldsymbol{z} & \boldsymbol{B} + \boldsymbol{B}^\top - \mathrm{tr}(\boldsymbol{B})\boldsymbol{I} \end{bmatrix} \tag{5}$$

where $\boldsymbol{z} = \left(B_{23} - B_{32}, B_{31} - B_{13}, B_{12} - B_{21}\right)^\top$, so that $\boldsymbol{W}\boldsymbol{q} = \lambda_{\max}\boldsymbol{q}$. This problem has been solved by a branch of Wahba's solvers, e.g. [8]–[10]. Recently, the problem has been revisited, generating two simplified approaches via symbolic results [11], [12]. New innovative research has also shown that orthonormalizing 3-D rotation can be performed with higher numerical accuracy in the 4-D space [13], [14]. Note that these methods are all numerical ones, including SVD and EIG, which requires Jacobi rotation or Householder transformation, that may not be simple to implement in circuits. A recent research item [15] shows that 3-D registration problem can be conducted algebraically, but the convergence rate is not so satisfactory for all cases, which is going to be refined and circuitized in this paper.

C. Contribution

This paper has the following main contributions:

- A new proportion-integral-derivative (PID)-like corrector has been proposed for faster convergence of rotation orthnormalization.
- We design a simple circuit synthesis of rotation orthonormalization problem for the first time and give its circuit prototype.
- Convergence analysis has been detailed, which shows the reliability of the designed circuit.

D. Outline

This paper is structured as follows: Section II contains our new theory and convergence analysis. Section III consists of experimental results and comparisons. Concluding remarks are drawn in Section IV.

II. Proposed Work

The Power method is an efficient method for solving eigenvector of associated with one matrix's maximum eigenvalue. In [15], the following system has been obtained to orthonormalize $\boldsymbol{B}$ by simplifying the Power method to solve eigenvalue problem $\boldsymbol{W}\boldsymbol{q} = \lambda_{\max}\boldsymbol{q}$:

$$\begin{aligned} \boldsymbol{h}_{x,k} &= \rho_{k-1}\left(\boldsymbol{h}_{x,k-1} + \boldsymbol{h}_{y,k-1} \times \boldsymbol{h}_{z,k-1}\right) \\ \boldsymbol{h}_{y,k} &= \rho_{k-1}\left(\boldsymbol{h}_{y,k-1} + \boldsymbol{h}_{z,k-1} \times \boldsymbol{h}_{x,k-1}\right) \\ \boldsymbol{h}_{z,k} &= \rho_{k-1}\left(\boldsymbol{h}_{z,k-1} + \boldsymbol{h}_{x,k-1} \times \boldsymbol{h}_{y,k-1}\right) \end{aligned} \tag{6}$$

for $k = 1, 2, \cdots$ and the quasi-normalization factor is $\rho_{k-1} = 2/\left(\alpha_{k-1} + 1\right)$ with $\boldsymbol{h}_{x,k}, \boldsymbol{h}_{y,k}, \boldsymbol{h}_{z,k}$ establishing three rows or columns of $\boldsymbol{H}_k$ and $\boldsymbol{H}_0 = \boldsymbol{B}/\max\left[\mathrm{abs}\left(\boldsymbol{B}\right)\right]$ and $\alpha_{k-1} = \left\|\boldsymbol{h}_{x,k-1}\right\|^2 + \left\|\boldsymbol{h}_{y,k-1}\right\|^2 + \left\|\boldsymbol{h}_{z,k-1}\right\|^2$. Such iterations in (6) are completely algebraic with addition, subtraction, multiplication, and devision only. Thus nonlinear operations like sin and cos are avoided, which are essential in mainstream SVD and EIG solutions. In (6), each row or column of the rotation matrix is orthogonal to others and is orthonormal. Therefore for limiting case we have $\boldsymbol{h}_x \perp \boldsymbol{h}_y \perp \boldsymbol{h}_z$ and $\boldsymbol{h}_x = \boldsymbol{h}_y \times \boldsymbol{h}_z$, $\boldsymbol{h}_y = \boldsymbol{h}_z \times \boldsymbol{h}_x$, $\boldsymbol{h}_z = \boldsymbol{h}_x \times \boldsymbol{h}_y$ and $\|\boldsymbol{h}_x\| = \|\boldsymbol{h}_y\| = \|\boldsymbol{h}_z\| = 1$ along with $\rho = 1/2$. Thus, for an improper rotation matrix, the cross product of two rows or columns gives a correction to the other previous column or row. Note that ρ_k is optimal only in the sense of the Power method solving $\boldsymbol{W}\boldsymbol{q} = \lambda_{\max}\boldsymbol{q}$. In this way, we can write out the error term between successive recursions (backward)

$$\begin{aligned} \Delta\boldsymbol{h}_{x,k-1} &= \boldsymbol{h}_{x,k-1} - \rho_{k-1}\left(\boldsymbol{h}_{x,k-1} + \boldsymbol{h}_{y,k-1} \times \boldsymbol{h}_{z,k-1}\right) \\ &= (1 - \rho_{k-1})\boldsymbol{h}_{x,k-1} - \rho_{k-1}\left(\boldsymbol{h}_{y,k-1} \times \boldsymbol{h}_{z,k-1}\right) \\ &= (1 - \rho_{k-1})\boldsymbol{h}_{x,k-1} + \rho_{k-1}\left(\boldsymbol{h}_{z,k-1} \times \boldsymbol{h}_{y,k-1}\right) \\ \Delta\boldsymbol{h}_{y,k-1} &= (1 - \rho_{k-1})\boldsymbol{h}_{y,k-1} + \rho_{k-1}\left(\boldsymbol{h}_{x,k-1} \times \boldsymbol{h}_{z,k-1}\right) \\ \Delta\boldsymbol{h}_{z,k-1} &= (1 - \rho_{k-1})\boldsymbol{h}_{z,k-1} + \rho_{k-1}\left(\boldsymbol{h}_{y,k-1} \times \boldsymbol{h}_{x,k-1}\right). \end{aligned} \tag{7}$$

We can clear see from (7) that the error term is presented in a complementary-filter form with equalizing factor of ρ_{k-1}; while in these filters, the correction terms are $\boldsymbol{h}_{z,k-1} \times \boldsymbol{h}_{y,k-1}$, $\boldsymbol{h}_{x,k-1} \times \boldsymbol{h}_{z,k-1}$ and $\boldsymbol{h}_{y,k-1} \times \boldsymbol{h}_{x,k-1}$ respectively. The structure of (7) provides another intuitive convergence analysis as follows, taking $\boldsymbol{h}_x$ as an example

- When $0 < \rho_{k-1} < 1$, we have $\alpha_{k-1} > 1$. In this way $0 < 1 - \rho_{k-1} < 1$ then $\boldsymbol{h}_{z,k-1} \times \boldsymbol{h}_{y,k-1}$ approaches $-\boldsymbol{h}_{x,k-1}$ and eventually $\Delta\boldsymbol{h}_x = \boldsymbol{0}$.
- When $1 \le \rho_{k-1} \le 2$, we have $-1 \le 1 - \rho_{k-1} \le 0$. The error term can be formulated as $\Delta\boldsymbol{h}_{x,k-1} = \beta_{k-1}\left(-\boldsymbol{h}_{x,k-1}\right) + (1 + \beta_{k-1})\left(\boldsymbol{h}_{z,k-1} \times \boldsymbol{h}_{y,k-1}\right)$ in which $0 \le \beta_{k-1} = \rho_{k-1} - 1 \le 1$. The convergence of this case can be cast into the first case, leading to the limiting result that $\Delta\boldsymbol{h}_x = \boldsymbol{0}$.

This convergence analysis shows that the recursion is globally convergent for arbitrary composition of $\boldsymbol{h}_x, \boldsymbol{h}_y$ and $\boldsymbol{h}_z$. It also gives an intuitive reason that a renowned cross-product based complementary filter in [16] is conditionally convergent. Iterations of (6) usually converge within 10 steps. The proposed recursion can be expedited by introducing PID-like parameters. Consider a PD corrector with proportional

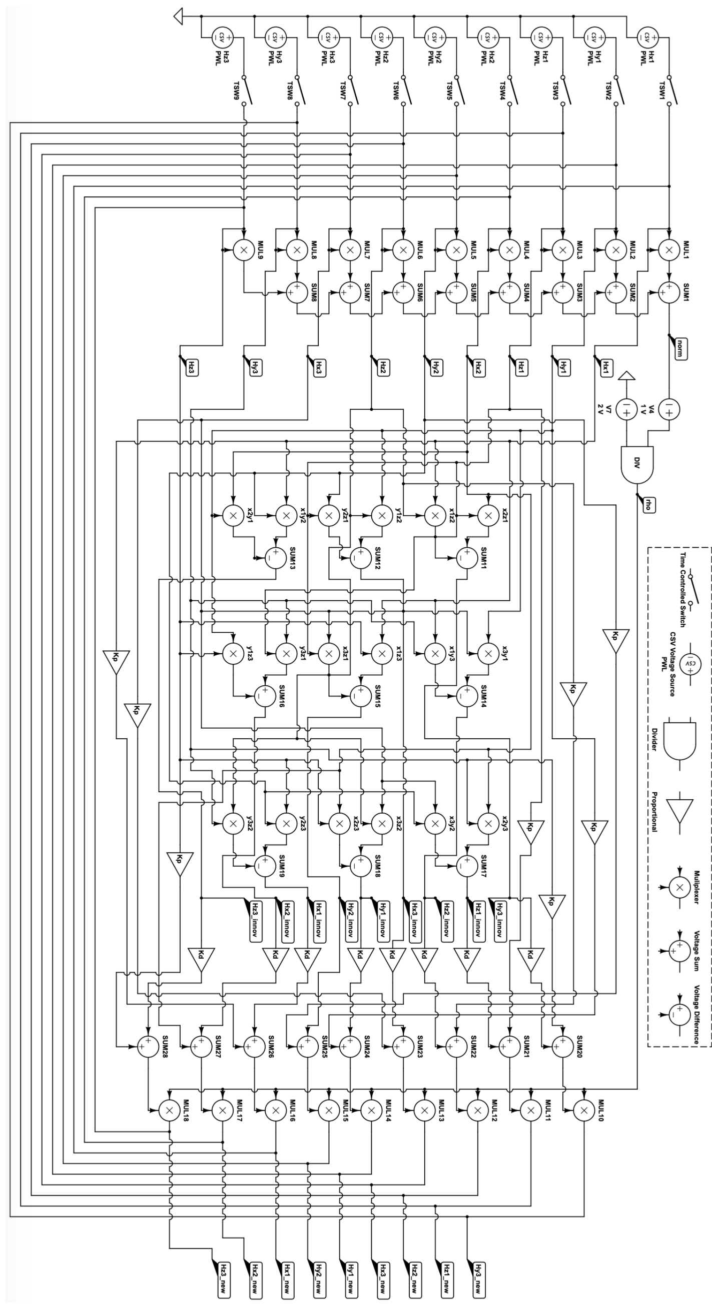

Fig. 1. The designed simple algebraic circuit prototype for 3-D rotation orthonormalization.

parameters of K_P and K_D. To make the following iteration also satisfies the global convergence analysis shown above:

$$\begin{aligned} \boldsymbol{h}_{x,k} &= \rho_{k-1}\left(K_P \boldsymbol{h}_{x,k-1} + K_D \boldsymbol{h}_{y,k-1} \times \boldsymbol{h}_{z,k-1}\right) \\ \boldsymbol{h}_{y,k} &= \rho_{k-1}\left(K_P \boldsymbol{h}_{y,k-1} + K_D \boldsymbol{h}_{z,k-1} \times \boldsymbol{h}_{x,k-1}\right) \\ \boldsymbol{h}_{z,k} &= \rho_{k-1}\left(K_P \boldsymbol{h}_{z,k-1} + K_D \boldsymbol{h}_{x,k-1} \times \boldsymbol{h}_{y,k-1}\right) \end{aligned} \tag{8}$$

with $\rho_{k-1} = 2/\left(\left\|\boldsymbol{h}_{x,k-1}\right\|^2 + \left\|\boldsymbol{h}_{y,k-1}\right\|^2 + \left\|\boldsymbol{h}_{z,k-1}\right\|^2 + 1\right)$, the steady-state equality must hold $\left\|\boldsymbol{h}_k\right\| = 1$, i.e.

$$1 = \rho_k^2 \left(\begin{array}{r} K_P^2 \left\|\boldsymbol{h}_{x,k}\right\|^2 + K_D^2 \left\|\boldsymbol{h}_{y,k} \times \boldsymbol{h}_{z,k}\right\|^2 + \\ 2K_P K_D \boldsymbol{h}_{x,k}^\top \boldsymbol{h}_{y,k} \times \boldsymbol{h}_{z,k} \end{array} \right) \tag{9}$$

so we have $(K_P + K_D)^2 = 4$ and then $K_D = 2 - K_P$. (8) is globally convergent. The reason is simple: such selection of K_P and K_D does not break the inequality condition shown previously. Here K_P and K_D are not exactly the correction parameters that appear in PID observers and controllers. Rather, they are presented in a similar fashion for the fast convergence of the recursion (6). It is also noted that the new structure (8) does not require an additional type of algebraic operations, so the circuitization will be simple and efficient. Denoting $\boldsymbol{h}_x = (x_1, x_2, x_3)^\top$, $\boldsymbol{h}_y = (y_1, y_2, y_3)^\top$ and $\boldsymbol{h}_z = (z_1, z_2, z_3)^\top$, using the cross-product formula $\boldsymbol{h}_x \times \boldsymbol{h}_y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)^\top$, the first simple fast 3-D rotation orthonormalization circuit is shown in Fig. 1.

III. Experimental Results

The circuit shown in Fig. 1 is discrete and can be implemented using digital addition, subtraction, multiplication and division components. If analog signals are taken into account, (8) should be converted into an approximate continuous form. However, analog devices are highly sensitive to temperature, input bandwidth and working frequency. Thus, in this paper, the circuit (8) is implemented on an FPGA platform in a discrete manner. We first show some characteristics of the new algorithm compared with representatives. For common orthonormalization tasks, we generate a set of 10000 initial $\boldsymbol{B}$ matrices whose column are subject to Gaussian distribution with covariance of $\boldsymbol{I}$. Using different parameters of K_P and K_D satisfying $K_P + K_D = 2$, the mean convergence performance of the designed new algorithm (8) is shown in Fig. 2. The algorithm `Original` denotes the one that uses $K_P = K_D = 1$ in [15]. The reference rotation is denoted as $\boldsymbol{R}_{\text{true}}$ which has been generated using SVD orthonormalization [6]. We can see that there are some combinations of parameters that outperform the original algorithm with slightly faster convergence speed. For most algorithms, they reach satisfactory orthonormalization within 10 iterations. This indicates that this group of $\boldsymbol{B}$ matrices are easy to be orthonormalized.

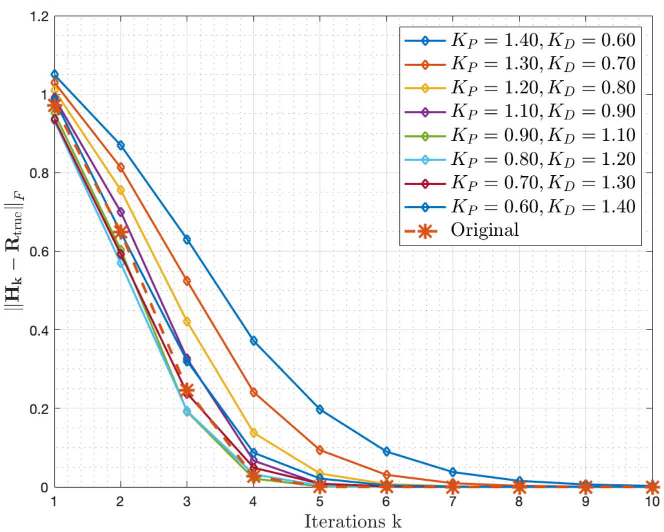

Fig. 2. Convergence performance of orthonormalization algorithms with different parameters for $\boldsymbol{B}$ matrices with columns subject to Gaussian distribution with covariance of $\boldsymbol{I}$.

A set of 100000 uniformly random rotation matrices are generated using [17]. We contaminate these rotation matrices in there columns by taking white Gaussian noise $\boldsymbol{\xi}$ into account, such that $\boldsymbol{\xi} \sim \mathcal{N}(\boldsymbol{0}, \sigma^2 \boldsymbol{I})$, where σ is the standard deviation. We perform an experiment with different σ. For each σ, we generate 100000 samples to evaluate the mean errors. The root mean squared errors (RMSEs) are in Table I. We can see that using $k = 10$ iterations, for higher σ, the one with $K_P = 0.8$ behaves better than else ones. For those rotation matrices are almost not noisy ($\sigma = 10^{-3}$), the original one conducts better orthonormalization and the differences between algorithms are negligible.

TABLE I
Mean RMSE Stats of Various Algorithms for 3-D Rotation Orthonormalization ($k = 10$)

Algorithms $\sigma = 10^{-3}$	$\|\boldsymbol{H}_k - \boldsymbol{R}_{\text{true}}\|_F$	abs$[\det(\boldsymbol{H}_k) - 1]$
Proposed ($K_P = 1.2$)	0.00161837959101	$7.04277902663 \times 10^{-14}$
Proposed ($K_P = 0.8$)	**0.00161837959099**	$3.01829250482 \times 10^{-14}$
Original	**0.00161837959099**	$\mathbf{2.90934381900 \times 10^{-14}}$
Algorithms $\sigma = 10^{-2}$		
Proposed ($K_P = 1.2$)	0.0164549084574	$7.10930847525 \times 10^{-13}$
Proposed ($K_P = 0.8$)	**0.0164549084567**	$\mathbf{2.14273043753 \times 10^{-13}}$
Original	0.0164552370284	$2.87822521370 \times 10^{-13}$
Algorithms $\sigma = 10^{-1}$		
Proposed ($K_P = 1.2$)	0.156878515087	$9.64848179130 \times 10^{-13}$
Proposed ($K_P = 0.8$)	**0.156878515061**	$\mathbf{9.44470168739 \times 10^{-13}}$
Original	0.156878554842	$9.52028619368 \times 10^{-13}$

For those matrices with unbalanced elements, orthonormalization becomes harder. What we are going to show next is an example

$$\boldsymbol{B} = \begin{pmatrix} 0.001 & 0.002 & 0.003 \\ 0.004 & -0.005 & -0.001 \\ 0.009 & -0.007 & 99.8 \end{pmatrix} \tag{10}$$

which owns a major entry in the last row and column. Its true orthonormalized one is

$$\boldsymbol{R}_{\text{true}} = \begin{pmatrix} -0.8945 & -0.4472 & 7.936e-5 \\ 0.4472 & -0.8945 & -0.0001 \\ 0.0001 & -6.5650e-5 & 0.9999 \end{pmatrix}. \tag{11}$$

Several tests with different parameters of K_P and K_D are also arranged; those results are shown in Fig. 3. For initial iterations of $k = 1 \sim 10$, the orthonormalization errors are large, indicating a stage of equalizing the largest number in $\boldsymbol{B}$. After $k = 10$, all the algorithms converge rapidly. Among all candidates, the one with $K_P = 0.8, K_D = 1.2$ converges the fastest. This shows that better parameters indeed improve the convergence performance of the orthonormalization.

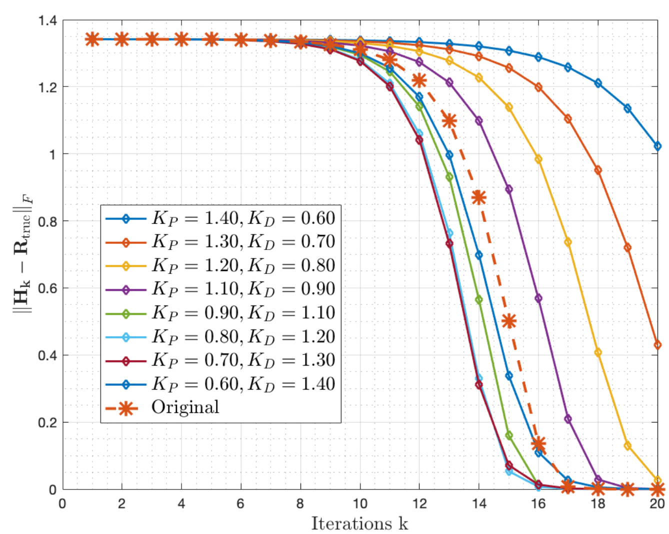

Fig. 3. Convergence performance of orthonormalization algorithms with different parameters for $\boldsymbol{B}$ matrix shown in (10).

From (10), we have $\det(\boldsymbol{B}) = -0.001297$. The SVD of $\boldsymbol{B} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top$ indicates that $\det(\boldsymbol{U}\boldsymbol{V}) = -1$. Therefore, when using SVD for orthonormalization, the result is $\boldsymbol{R} = \boldsymbol{U}\mathrm{diag}\,[1, 1, -1]\,\boldsymbol{V}^\top$. For the proposed method, using different K_P and K_D, it is able for us to illustrate the evolution of determinant of $\boldsymbol{H}_k$ (see Fig. 4). The results show that even for extreme case like (10), the developed algorithm is able to orthonormalize $\boldsymbol{B}$ with negative determinant exactly into $SO(3)$ such that $\det(\boldsymbol{H}_k) = 1$ when $k \to \infty$. Detailed reason has been shown in [15] that when $k \to \infty$, it theoretically follows that $\det(\boldsymbol{H}_k) = 1$. If we do not perform a key step $\boldsymbol{H}_0 = \boldsymbol{B}/\max[\mathrm{abs}(\boldsymbol{B})]$ as initialization, the convergence inequalities in [15] would not hold. Then (6) will only be valid for non-$SO(3)$ orthonormalization, which may result in $\det(\boldsymbol{H}_k) = -1$.

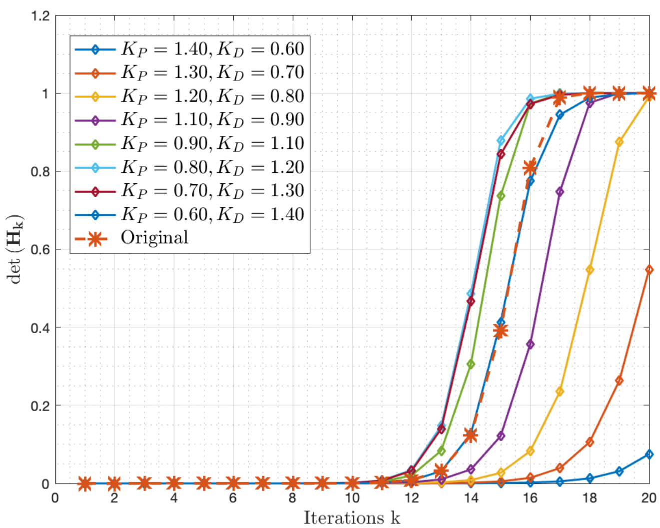

Fig. 4. Convergence performance of $\det(\boldsymbol{H}_k)$ with different parameters for $\boldsymbol{B}$ matrix shown in (10).

The proposed digital circuit is implemented with VerilogHDL language and deployed on a Xilinx Kintex-7 FPGA chip with a part number of XC7K325T. The implementation reports indicate that this design occupied very limited resources with 17 registers, 2 digital slices and 2 DSP units. The power consumption of this functional block is 6 mW, which was an admirable result, but it can be significantly reduced by using an application specific integrated circuit (ASIC) implementation for high-end applications. Regarding the implementation of previous popular algorithms like SVD and EIG, the computational resources are much more employed. For the target platform, the SVD takes 127 digital slices for Householder transform. Note that the SVD algorithm has been simplified to its best, e.g. important sub-blocks like coordinate rotation digital computer (CORDIC, [18]) has been efficiently provided by the internal library. SVD consumes a power level of 297 mW, which is much higher than that of the developed method. This shows that the proposed method is more suitable for compact, low-cost and low-power industrial electronics.

IV. Conclusion

The rotation orthonormalization problem has been revisited in this paper. For the first time in the community, we design a very simple and efficient algorithm that can be easily implemented on various platforms. It is shown that the introduction of a PID-corrector can significantly improve the convergence speed without loss of accuracy. The proposed algorithm is also more efficient for those orthonormalization tasks with extremely unbalanced matrix entries. Since only four elementary algebraic operations are involved, the designed globally convergent algorithm circuitized has been tested successful on Xilinx FPGA platform. Future efforts need to be paid to simplify a related system design for ultra low-cost and low-power ASIC.

Acknowledgement

We would like to thank Prof. Gongmin Yan from Northwestern Polytechnic University, Xi'an, China for his constructive comments.

References

[1] A. De Vita, G. D. Licciardo, A. Femia, L. Di Benedetto, A. Rubino, and D. Pau, "Embeddable Circuit for Orientation Independent Processing in Ultra Low-Power Tri-axial Inertial Sensors," *IEEE Trans. Circuits Syst. II Express Briefs*, vol. 67, no. 6, pp. 1–1, 2019.

[2] S. Amini and D. A. Johns, "A Flexible Charge-Balanced Ratiometric Open-Loop Readout System for Capacitive Inertial Sensors," *IEEE Trans. Circuits Syst. II Express Briefs*, vol. 62, no. 4, pp. 317–321, 2015.

[3] R. Hartley and A. Zisserman, *Multiple View Geometry in Computer Vision*. Cambridge University Press, 2003.

[4] I. Y. Bar-Itzhack and K. A. Fegley, "Orthogonalization Techniques of a Direction Cosine Matrix," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-5, no. 5, pp. 798–804, 1969.

[5] X. Huang, J. Zhang, Q. Wu, L. Fan, and C. Yuan, "A Coarse-to-Fine Algorithm for Matching and Registration in 3D Cross-Source Point Clouds," *IEEE Trans. Circuits Syst. Video Technol.*, vol. 28, no. 10, pp. 2965–2977, 2018.

[6] K. S. Arun, T. S. Huang, and S. D. Blostein, "Least-Squares Fitting of Two 3-D Point Sets," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. PAMI-9, no. 5, pp. 698–700, 1987.

[7] S. Umeyama, "Least-Squares Estimation of Transformation Parameters Between Two Point Patterns," *IEEE Trans. Pattern Anal. Mach. Intell.*, pp. 376–380, 1991.

[8] M. D. Shuster and S. D. Oh, "Three-Axis Attitude Determination from Vector Observations," *J. Guid. Control. Dyn.*, vol. 4, no. 1, pp. 70–77, 1981.

[9] F. L. Markley, "Attitude Determination Using Vector Observations - A Fast Optimal Matrix Algorithm," *J. Astronaut. Sci.*, vol. 41, no. 2, pp. 261–280, 1993.

[10] D. Mortari, F. L. Markley, and P. Singla, "Optimal Linear Attitude Estimator," *J. Guid. Control. Dyn.*, vol. 30, no. 6, pp. 1619–1627, 2007.

[11] J. Wu, M. Liu, Z. Zhou, and R. Li, "Fast Symbolic 3-D Registration Solution," *IEEE Trans. Autom. Sci. Eng.*, vol. 17, no. 2, pp. 761–770, 2020.

[12] S. Sarabandi, A. Shabani, J. M. Porta, and F. Thomas, "On Closed-Form Formulas for the 3-D Nearest Rotation Matrix Problem," *IEEE Trans. Robot.*, pp. 1–7, 2020.

[13] S. Sarabandi, A. Perez-Gracia, and F. Thomas, "On Cayley's Factorization with an Application to the Orthonormalization of Noisy Rotation Matrices," *Adv. Appl. Clifford Algebr.*, vol. 29, no. 3, pp. 1–16, 2019.

[14] J. Wu, Y. Sun, M. Wang, and M. Liu, "Hand-Eye Calibration: 4-D Procrustes Analysis Approach," *IEEE Trans. Instrum. Meas.*, vol. 69, no. 6, pp. 2966–2981, 2019.

[15] J. Wu, "Rigid 3D Registration: A Simple Method Free of SVD and Eigen-Decomposition," *IEEE Trans. Instrum. Meas.*, pp. 1–1, 2020.

[16] R. Mahony, T. Hamel, and J. M. Pflimlin, "Nonlinear Complementary Filters on the Special Orthogonal Group," *IEEE Trans. Automat. Contr.*, vol. 53, no. 5, pp. 1203–1218, 2008.

[17] K. Shoemake, "Uniform Random Rotations," *Graphics Gems III*, pp. 124–132, 1992.

[18] H. Chen, K. Cheng, Z. Lu, Y. Fu, and L. Li, "Hyperbolic CORDIC-Based Architecture for Computing Logarithm and Its Implementation," *IEEE Trans. Circuits Syst. II Express Briefs*, 2020.